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Supplement to

A NONCONFORMING MIXED MULTIGRID METHOD FOR THE PURE TRACTION PROBLEM IN PLANAR LINEAR ELASTICITY

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This supplement contains the proofs of the theorems stated in §5.

Lemma S.1. *There holds* $(\underset{\sim}{v}^\sharp, q^\sharp) = P_k^{k-1}(\underset{\sim}{y} - \underset{\sim}{y}_m, z - z_m)$.

Proof. Note that $(\underset{\sim}{v}^\sharp, q^\sharp)$ and $P_k^{k-1}(\underset{\sim}{y} - \underset{\sim}{y}_m, z - z_m)$ both belong to $\underset{\sim}{V}_{k-1}^\perp \times Q_{k-2}$. Given any $(\underset{\sim}{v}, q) \in \underset{\sim}{V}_{k-1}^\perp \times Q_{k-2}$, it follows from (3.4) and (3.11) that

$$\begin{aligned}
\mathcal{B}_{k-1}\left((\underset{\sim}{v}^\sharp, q^\sharp), (\underset{\sim}{v}, q)\right) &= \left(B_{k-1}^\perp(\underset{\sim}{v}^\sharp, q^\sharp), (\underset{\sim}{v}, q)\right)_{k-1} \\
&= \left(I_k^{k-1} B_k(\underset{\sim}{y} - \underset{\sim}{y}_m, z - z_m), (\underset{\sim}{v}, q)\right)_{k-1} \\
&= \left(B_k(\underset{\sim}{y} - \underset{\sim}{y}_m, z - z_m), I_{k-1}^k(\underset{\sim}{v}, q)\right)_k \\
&= \mathcal{B}_k\left((\underset{\sim}{y} - \underset{\sim}{y}_m, z - z_m), I_{k-1}^k(\underset{\sim}{v}, q)\right) \\
&= \mathcal{B}_{k-1}\left(P_k^{k-1}(\underset{\sim}{y} - \underset{\sim}{y}_m, z - z_m), (\underset{\sim}{v}, q)\right).
\end{aligned}$$

The lemma now follows from Proposition 2.7. □

Let the kth-level relaxation operator R_k be defined by

$$R_k := I - \frac{1}{\Lambda_k}\left(B_k^\perp\right)^2. \tag{S.1}$$

(Recall that $\Lambda_k := C\, h_k^{-4}$ dominates the spectral radius of $(B_k^\perp)^2$.)

Note that by the definition of the mesh-dependent norms we have

$$|\!|\!| R_k(\underset{\sim}{v}, q) |\!|\!|_{s,k} \le |\!|\!| (\underset{\sim}{v}, q) |\!|\!|_{s,k} \quad \forall (\underset{\sim}{v}, q) \in \underset{\sim}{V}_k^\perp \times Q_{k-1} \text{ and } s \in \mathbb{R}. \tag{S.2}$$

From the smoothing step (4.2), we obtain

$$(\underset{\sim}{y} - \underset{\sim}{y}_m, z - z_m) = R_k^m(\underset{\sim}{y} - \underset{\sim}{y}_0, z - z_0). \tag{S.3}$$

Combining Lemma S.1 and (S.3), we have the following relation between the initial error and the final error of the two-grid algorithm:

$$(\underset{\sim}{y} - \underset{\sim}{y}^\sharp, z - z^\sharp) = (I - I_{k-1}^k P_k^{k-1}) R_k^m(\underset{\sim}{y} - \underset{\sim}{y}_0, z - z_0). \tag{S.4}$$

The effect of R_k^m is measured by the following lemma on the smoothing property.

Lemma S.2 (Smoothing Property). *There exists a positive constant C such that*

$$\text{(S.5)}\qquad |\!|\!| R_k^m(\utilde{v},q) |\!|\!|_{2,k} \le C\, h_k^{-2}\, m^{-1/2}\, |\!|\!| (\utilde{v},q) |\!|\!|_{0,k} \quad \forall (\utilde{v},q) \in \utilde{V}_k^{\perp} \times Q_{k-1}.$$

Proof. Let $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{n_k}$ be the eigenvalues of $(B_k^{\perp})^2$ and $\boldsymbol{\eta}_i := (\utilde{\phi}_i, \psi_i)$, $1 \le i \le n_k$ be the corresponding eigenvectors satisfying the orthonormal relation $(\boldsymbol{\eta}_i, \boldsymbol{\eta}_j)_k = \delta_{ij}$. We can write $(\utilde{v},q) = \sum_{i=1}^{n_k} \nu_i \boldsymbol{\eta}_i$. Hence,

$$\begin{aligned} R_k^m(\utilde{v},q) &= \left(I - \frac{1}{\Lambda_k}(B_k^{\perp})^2\right)^m (\utilde{v},q) \\ &= \sum_{i=1}^{n_k} \left(1 - \frac{\lambda_i}{\Lambda_k}\right)^m \nu_i \boldsymbol{\eta}_i. \end{aligned}$$

Hence,

$$\begin{aligned} |\!|\!| R_k^m(\utilde{v},q) |\!|\!|_{2,k}^2 &= \sum_{i=1}^{n_k} \left(1 - \frac{\lambda_i}{\Lambda_k}\right)^{2m} \lambda_i \nu_i^2 \\ &= \Lambda_k \left\{ \sum_{i=1}^{n_k} \left(1 - \frac{\lambda_i}{\Lambda_k}\right)^{2m} \left(\frac{\lambda_i}{\Lambda_k}\right) \nu_i^2 \right\} \\ &= \Lambda_k \left\{ \sup_{0 \le x \le 1} (1-x)^{2m} x \right\} \sum_{i=1}^{n_k} \nu_i^2 \\ &\le C\, h_k^{-4}\, m^{-1}\, |\!|\!| (\utilde{v},q) |\!|\!|_{0,k}^2. \quad \square \end{aligned}$$

The following lemma on the approximation property measures the effect of the operators $I - I_{k-1}^k P_k^{k-1}$.

Lemma S.3 (Approximation Property). *There exists a positive constant C such that for $k > 1$*

$$\text{(S.6)}\qquad |\!|\!| (I - I_{k-1}^k P_k^{k-1})(\utilde{v},q) |\!|\!|_{0,k} \le C\, h_k^2\, |\!|\!| (\utilde{v},q) |\!|\!|_{2,k} \quad \forall (\utilde{v},q) \in \utilde{V}_k^{\perp} \times Q_{k-1}.$$

Proof. Given $(\utilde{v},q) \in \utilde{V}_k^{\perp} \times Q_{k-1}$, let $(\utilde{\eta},\tau) = P_k^{k-1}(\utilde{v},q)$. Hence,

$$\text{(S.7)}\qquad (I - I_{k-1}^k P_k^{k-1})(\utilde{v},q) = (\utilde{v} - \utilde{P}_k\utilde{\eta}, q - \tau).$$

Recall that $|\!|\!| (\utilde{v} - \utilde{P}_k\utilde{\eta}, q-\tau) |\!|\!|_{0,k}^2 = \|\utilde{v} - \utilde{P}_k\utilde{\eta}\|_{\utilde{L}^2(\Omega)}^2 + h_k^2 \|q-\tau\|_{L^2(\Omega)}^2$. We shall estimate $h_k\|q-\tau\|_{L^2(\Omega)}$ and $\|\utilde{v} - \utilde{P}_k\utilde{\eta}\|_{\utilde{L}^2(\Omega)}$ by two duality arguments.

Let $(\utilde{v}_k^0, q_k^0) \in \utilde{V}_k^{\perp} \times Q_{k-1}$ satisfy

$$\text{(S.8)}\qquad \mathcal{B}_k\left((\utilde{v}_k^0, q_k^0), (\utilde{v}', q')\right) = h_k^2 \int_\Omega (q-\tau)\, q'\, dx \quad \forall (\utilde{v}', q') \in \utilde{V}_k^{\perp} \times Q_{k-1},$$

and $(\utilde{v}_{k-1}^0, q_{k-1}^0) \in \utilde{V}_{k-1}^{\perp} \times Q_{k-2}$ satisfy

$$\text{(S.9)}\qquad \mathcal{B}_{k-1}\left((\utilde{v}_{k-1}^0, q_{k-1}^0), (\utilde{v}', q')\right) = h_k^2 \int_\Omega (q-\tau)\, q'\, dx \quad \forall (\utilde{v}', q') \in \utilde{V}_{k-1}^{\perp} \times Q_{k-2}.$$

It follows from (S.8), (S.9), the definition of $(\utilde{\eta},\tau)$, (3.11), (3.9) and Lemma 3.7 that

$$\begin{aligned} h_k^2 \|q-\tau\|_{L^2(\Omega)}^2 &= h_k^2 \int_\Omega (q-\tau)\, q\, dx - h_k^2 \int_\Omega (q-\tau)\, \tau\, dx \\ &= \mathcal{B}_k\big((\utilde{v}_k^0, q_k^0), (\utilde{v},q)\big) - \mathcal{B}_{k-1}\big((\utilde{v}_{k-1}^0, q_{k-1}^0), (\utilde{\eta},\tau)\big) \\ &= \mathcal{B}_k\big((\utilde{v}_k^0, q_k^0), (\utilde{v},q)\big) - \mathcal{B}_{k-1}\big((\utilde{v}_{k-1}^0, q_{k-1}^0), P_k^{k-1}(\utilde{v},q)\big) \\ &= \mathcal{B}_k\big((\utilde{v}_k^0, q_k^0) - I_{k-1}^k(\utilde{v}_{k-1}^0, q_{k-1}^0), (\utilde{v},q)\big) \\ &\le |\!|\!| (\utilde{v}_k^0, q_k^0) - I_{k-1}^k(\utilde{v}_{k-1}^0, q_{k-1}^0) |\!|\!|_{0,k}\, |\!|\!| (\utilde{v},q) |\!|\!|_{2,k} \\ &\le C\, h_k^3\, \|q-\tau\|_{L^2(\Omega)}\, |\!|\!| (\utilde{v},q) |\!|\!|_{2,k}. \end{aligned}$$

Therefore, we have

$$\text{(S.10)}\qquad h_k \|q-\tau\|_{L^2(\Omega)} \le C\, h_k^2\, |\!|\!| (\utilde{v},q) |\!|\!|_{2,k}.$$

We now turn to the estimate of $\|\utilde{v} - \utilde{P}_k\utilde{\eta}\|_{\utilde{L}^2(\Omega)}$. Since $\utilde{v} - \utilde{P}_k\utilde{\eta} \in \utilde{L}_{\perp}^2$, there exists a unique $(\utilde{\zeta},\xi) \in \utilde{H}_{\perp}^2 \times H^1(\Omega)$ satisfying

$$\text{(S.11)}\qquad \mathcal{B}\left((\utilde{\zeta},\xi), (\utilde{v}', q')\right) = \int_\Omega (\utilde{v} - \utilde{P}_k\utilde{\eta}) \cdot \utilde{v}'\, dx \quad \forall (\utilde{v}', q') \in \utilde{H}_{\perp}^1(\Omega) \times L^2(\Omega).$$

Let $(\utilde{\zeta}_k, \xi_k) \in \utilde{V}_k^{\perp} \times Q_{k-1}$ satisfy

$$\text{(S.12)}\qquad \mathcal{B}_k\left((\utilde{\zeta}_k, \xi_k), (\utilde{v}', q')\right) = \int_\Omega (\utilde{v} - \utilde{P}_k\utilde{\eta}) \cdot \utilde{v}'\, dx \quad \forall (\utilde{v}', q') \in \utilde{V}_k^{\perp} \times Q_{k-1},$$

and $(\utilde{\zeta}_{k-1}, \xi_{k-1}) \in \utilde{V}_{k-1}^{\perp} \times Q_{k-2}$ satisfy

$$\text{(S.13)}\qquad \mathcal{B}_{k-1}\left((\utilde{\zeta}_{k-1}, \xi_{k-1}), (\utilde{v}', q')\right) = \int_\Omega (\utilde{v} - \utilde{P}_k\utilde{\eta}) \cdot \utilde{v}'\, dx \quad \forall (\utilde{v}', q') \in \utilde{V}_{k-1}^{\perp} \times Q_{k-2}.$$

The elliptic regularity estimate (1.20) implies that (recall that $\xi = \frac{\lambda}{2\mu} \operatorname{div} \utilde{\zeta}$)

$$\text{(S.14)}\qquad \|\utilde{\zeta}\|_{\utilde{H}^2(\Omega)} + \|\xi\|_{H^1(\Omega)} \le C\, \|\utilde{v} - \utilde{P}_k\utilde{\eta}\|_{\utilde{L}^2(\Omega)},$$

and the discretization error estimates in Theorems 2.9 and 2.10 imply that

$$
\begin{aligned}
\text{(S.15)}\qquad & \|\underset{\sim}{\zeta}_k - \underset{\sim}{\zeta}\|_{L^2(\Omega)} + \|\underset{\sim}{\zeta}_{k-1} - \underset{\sim}{\zeta}\|_{L^2(\Omega)} \\
& + h_k\|\xi_k - \xi\|_{L^2(\Omega)} + h_k\|\xi_{k-1} - \xi\|_{L^2(\Omega)} \\
& \le C\, h_k^2 \,\|\underset{\sim}{v} - P_k\underset{\sim}{\eta}\|_{L^2(\Omega)}.
\end{aligned}
$$

The interpolation error estimates (2.30), (2.31) and (S.14) imply that

$$
\begin{aligned}
\text{(S.16)}\qquad & \|\Pi_k^{\perp}\underset{\sim}{\zeta} - \underset{\sim}{\zeta}\|_{L^2(\Omega)} + \|\Pi_{k-1}^{\perp}\underset{\sim}{\zeta} - \underset{\sim}{\zeta}\|_{L^2(\Omega)} \\
& + h_k\|P_{k-1}\xi - \xi\|_{L^2(\Omega)} + h_k\|P_{k-2}\xi - \xi\|_{L^2(\Omega)} \\
& \le C\, h_k^2 \,\|\underset{\sim}{v} - P_k\underset{\sim}{\eta}\|_{L^2(\Omega)}.
\end{aligned}
$$

From the definition of $(\underset{\sim}{\eta}, \tau)$, (S.12), (S.13) and (3.11) we have

$$
\begin{aligned}
\text{(S.17)}\qquad & \|\underset{\sim}{v} - P_k\underset{\sim}{\eta}\|^2_{L^2(\Omega)} \\
&= \mathcal{B}_k\Big((\underset{\sim}{\zeta}_k, \xi_k), (\underset{\sim}{v}, q)\Big) - \mathcal{B}_k\Big((\underset{\sim}{\zeta}_k, \xi_k), I_{k-1}^k(\underset{\sim}{\eta}, \tau)\Big) \\
&= \mathcal{B}_k\Big((\underset{\sim}{\zeta}_k, \xi_k) - (\Pi_k^{\perp}\underset{\sim}{\zeta}, P_{k-1}\xi), (\underset{\sim}{v}, q)\Big) \\
&+ \mathcal{B}_k\Big((\Pi_k^{\perp}\underset{\sim}{\zeta}, P_{k-1}\xi) - I_{k-1}^k(\Pi_{k-1}^{\perp}\underset{\sim}{\zeta}, P_{k-2}\xi), (\underset{\sim}{v}, q)\Big) \\
&+ \mathcal{B}_k\Big(I_{k-1}^k(\Pi_{k-1}^{\perp}\underset{\sim}{\zeta}, P_{k-2}\xi), (\underset{\sim}{v}, q)\Big) - \int_\Omega (\underset{\sim}{v} - P_k\underset{\sim}{\eta}) \cdot P_k\underset{\sim}{\eta}\, dx \\
&= \mathcal{B}_k\Big((\underset{\sim}{\zeta}_k, \xi_k) - (\Pi_k^{\perp}\underset{\sim}{\zeta}, P_{k-1}\xi), (\underset{\sim}{v}, q)\Big) \\
&+ \mathcal{B}_k\Big((\Pi_k^{\perp}\underset{\sim}{\zeta}, P_{k-1}\xi) - I_{k-1}^k(\Pi_{k-1}^{\perp}\underset{\sim}{\zeta}, P_{k-2}\xi), (\underset{\sim}{v}, q)\Big) \\
&+ \mathcal{B}_{k-1}\Big((\Pi_{k-1}^{\perp}\underset{\sim}{\zeta}, P_{k-2}\xi) - (\underset{\sim}{\zeta}_{k-1}, \xi_{k-1}), P_k^{k-1}(\underset{\sim}{v}, q)\Big) \\
&+ \int_\Omega (\underset{\sim}{v} - P_k\underset{\sim}{\eta})(\underset{\sim}{\eta} - P_k\underset{\sim}{\eta})\, dx.
\end{aligned}
$$

Combining (3.9), (S.15), (S.16), (3.12), Lemma 3.4 (iii), Lemma 3.6, (S.14) and the definition of $(\underset{\sim}{\eta}, \tau)$, we have

$$
\begin{aligned}
\text{(S.18)}\qquad & \|\underset{\sim}{v} - P_k\underset{\sim}{\eta}\|^2_{L^2(\Omega)} \\
&\le |\!|\!|(\underset{\sim}{\zeta}_k, \xi_k) - (\Pi_k^{\perp}\underset{\sim}{\zeta}, P_{k-1}\xi)|\!|\!|_{0,k}\, |\!|\!|(\underset{\sim}{v}, q)|\!|\!|_{2,k} \\
&+ |\!|\!|(\Pi_k^{\perp}\underset{\sim}{\zeta}, P_{k-1}\xi) - I_{k-1}^k(\Pi_{k-1}^{\perp}\underset{\sim}{\zeta}, P_{k-2}\xi)|\!|\!|_{0,k}\, |\!|\!|(\underset{\sim}{v}, q)|\!|\!|_{2,k} \\
&+ |\!|\!|(\Pi_{k-1}^{\perp}\underset{\sim}{\zeta}, P_{k-2}\xi) - (\underset{\sim}{\zeta}_{k-1}, \xi_{k-1})|\!|\!|_{0,k-1}\, |\!|\!|P_k^{k-1}(\underset{\sim}{v}, q)|\!|\!|_{2,k-1} \\
&+ \|\underset{\sim}{v} - P_k\underset{\sim}{\eta}\|_{L^2(\Omega)} \|\underset{\sim}{\eta} - P_k\underset{\sim}{\eta}\|_{L^2(\Omega)} \\
&\le C\, h_k^2\, \|\underset{\sim}{v} - P_k\underset{\sim}{\eta}\|_{L^2(\Omega)} |\!|\!|(\underset{\sim}{v}, q)|\!|\!|_{2,k} \\
&+ \|\underset{\sim}{v} - P_k\underset{\sim}{\eta}\|_{L^2(\Omega)} \|\underset{\sim}{\eta} - P_k\underset{\sim}{\eta}\|_{L^2(\Omega)} \\
&\le C\, h_k^2\, \|\underset{\sim}{v} - P_k\underset{\sim}{\eta}\|_{L^2(\Omega)} \Big\{ |\!|\!|(\underset{\sim}{v}, q)|\!|\!|_{2,k} + |\!|\!|(\underset{\sim}{\eta}, \tau)|\!|\!|_{2,k-1} \Big\} \\
&\le C\, h_k^2\, \|\underset{\sim}{v} - P_k\underset{\sim}{\eta}\|_{L^2(\Omega)} |\!|\!|(\underset{\sim}{v}, q)|\!|\!|_{2,k}.
\end{aligned}
$$

Hence,

$$
\text{(S.19)}\qquad \|\underset{\sim}{v} - P_k\underset{\sim}{\eta}\|_{L^2(\Omega)} \le C\, h_k^2\, |\!|\!|(\underset{\sim}{v}, q)|\!|\!|_{2,k}.
$$

The lemma now follows from (S.10) and (S.19). □

The convergence of the two-grid algorithm is obtained by combining the smoothing and approximation properties.

Theorem S.4 (Convergence of the Two-Grid Algorithm). *There exists a positive constant C such that for $k > 1$*

$$
\text{(S.20)}\qquad |\!|\!|(\underset{\sim}{y} - \underset{\sim}{y}^{\sharp}, z - z^{\sharp})|\!|\!|_{0,k} \le C\, m^{-1/2}\, |\!|\!|(\underset{\sim}{y} - \underset{\sim}{y}_0, z - z_0)|\!|\!|_{0,k},
$$

where $(\underset{\sim}{y}, z)$ solves (4.1), $(\underset{\sim}{y}_0, z_0)$ is the initial guess, and $(\underset{\sim}{y}^{\sharp}, z^{\sharp})$ is the output of the two-grid algorithm. Therefore, for m sufficiently large, the two-grid algorithm is a contraction with contraction number bounded away from one, independent of k.

Proof. Using (S.4), Lemmas S.3 and S.2, we have

$$
\begin{aligned}
|\!|\!|(\underset{\sim}{y} - \underset{\sim}{y}^{\sharp}, z - z^{\sharp})|\!|\!|_{0,k} &= |\!|\!|(I - I_{k-1}^k P_k^{k-1}) R_k^m (\underset{\sim}{y} - \underset{\sim}{y}_0, z - z_0)|\!|\!|_{0,k} \\
&\le C\, h_k^2\, |\!|\!|R_k^m(\underset{\sim}{y} - \underset{\sim}{y}_0, z - z_0)|\!|\!|_{2,k} \\
&\le C\, m^{-1/2} |\!|\!|(\underset{\sim}{y} - \underset{\sim}{y}_0, z - z_0)|\!|\!|_{0,k}. \quad \square
\end{aligned}
$$

A perturbation argument then yields the following theorem on the convergence of the kth-level iteration.

Theorem S.5 (Convergence of the kth-Level Iteration). *There exists a positive constant C such that when the kth-level iteration is applied to (4.1), we have*

$$(S.21)\qquad |\!|\!|(y,z) - MG(k,(y_0,z_0),(w,r))|\!|\!|_{0,k} \le C\,m^{-1/2}\,|\!|\!|(y-y_0, z-z_0)|\!|\!|_{0,k},$$

provided that m is chosen to be large enough. Therefore, for m sufficiently large, the kth-level iteration is a contraction with contraction number bounded away from one, independent of k.

Proof. Let C_* be the maximum of the constants that appear in Lemma 3.4 (i), (3.12) and Theorem S.4. Then (S.21) holds for $C = 2C_*$ and $m \ge 16C_*^6$. The proof is by induction. Since the first-level iteration is a direct solver, (S.21) clearly holds for $k = 1$. Assume (S.21) holds for $k-1$ ($k > 1$). Then from §4 and the definition of $(y^\sharp, z^\sharp)$,

$$(y,z) - MG(k,(y_0,z_0),(w,r)) = (y - y^\sharp, z - z^\sharp) + I_{k-1}^k\left[(v^\sharp - v_2, q^\sharp - q_2)\right].$$

Therefore, Theorem S.4 and Lemma 3.4 (i) imply that

$$(S.22)\qquad \begin{aligned} &|\!|\!|(y,z) - MG(k,(y_0,z_0),(w,r))|\!|\!|_{0,k}\\ &\quad\le C_* m^{-1/2}|\!|\!|(y-y_0, z-z_0)|\!|\!|_{0,k} + C_*|\!|\!|(v^\sharp - v_2, q^\sharp - q_2)|\!|\!|_{0,k-1}.\end{aligned}$$

Recall that $(v^\sharp, q^\sharp)$ is the exact solution of

$$(S.23)\qquad B_{k-1}^{\perp}(v,q) = I_k^{k-1}\left((w,r) - B_k(y_m, z_m)\right),$$

and (v_2, q_2) is the approximate solution of (S.23) obtained by using the $(k-1)$-level iteration twice with initial guess $(0,0)$. Hence, it follows from the induction hypothesis, Lemma S.1, (3.12), (S.3) and (S.2) that

$$(S.24)\qquad \begin{aligned} |\!|\!|(v^\sharp - v_2, q^\sharp - q_2)|\!|\!|_{0,k-1} &\le 4C_*^2 m^{-1}|\!|\!|(v^\sharp, q^\sharp)|\!|\!|_{0,k-1}\\ &= 4C_*^2 m^{-1}|\!|\!|P_k^{k-1}(y - y_m, z - z_m)|\!|\!|_{0,k-1}\\ &\le 4C_*^3 m^{-1}|\!|\!|(y - y_m, z - z_m)|\!|\!|_{0,k}\\ &= 4C_*^3 m^{-1}|\!|\!|R_k^m(y - y_0, z - z_0)|\!|\!|_{0,k}\\ &\le 4C_*^3 m^{-1}|\!|\!|(y - y_0, z - z_0)|\!|\!|_{0,k}.\end{aligned}$$

Combining (S.22) and (S.24), we have by the condition on m that

$$(S.25)\qquad \begin{aligned} &|\!|\!|(y,z) - MG(k,(y_0,z_0),(w,r))|\!|\!|_{0,k}\\ &\quad\le \left(C_* m^{-1/2} + 4C_*^4 m^{-1}\right)|\!|\!|(y-y_0, z-z_0)|\!|\!|_{0,k}\\ &\quad\le 2\,C_*\,m^{-1/2}\,|\!|\!|(y-y_0, z-z_0)|\!|\!|_{0,k}. \quad\square\end{aligned}$$

We finally turn to the convergence of the full multigrid algorithm. First we establish the following lemma.

Lemma S.6. *There exists a positive constant C such that*

$$(S.26)\qquad |\!|\!|(u_k, p_k) - I_{k-1}^k(u_{k-1}, p_{k-1})|\!|\!|_{0,k} \le C\,h_k^2\left\{\|f\|_{L^2(\Omega)} + \sum_{i=1}^n \|g_i\|_{H^{1/2}(\Gamma_i)}\right\}.$$

Here, (u_k, p_k) and (u_{k-1}, p_{k-1}) are the exact solutions of the discretized problem (1.26) corresponding to the triangulations $\mathcal{T}_k$ and $\mathcal{T}_{k-1}$, respectively.

Proof. Using (1.20), (1.31), (2.30), (2.31), Theorems 2.9 and 2.10, (3.8) and Lemma 3.4, we have

$$\begin{aligned} &|\!|\!|(u_k, p_k) - I_{k-1}^k(u_{k-1}, p_{k-1})|\!|\!|_{0,k}\\ &\le |\!|\!|(u_k - \Pi_k^{\perp}u, p_k - P_{k-1}p)|\!|\!|_{0,k} + |\!|\!|(\Pi_k^{\perp}u, P_{k-1}p) - I_{k-1}^k(\Pi_{k-1}^{\perp}u, P_{k-2}p)|\!|\!|_{0,k}\\ &\quad + |\!|\!|I_{k-1}^k(\Pi_{k-1}^{\perp}u - u_{k-1}, P_{k-2}p - p_{k-1})|\!|\!|_{0,k}\\ &\le C\,h_k^2\left\{\|f\|_{L^2(\Omega)} + \sum_{i=1}^n \|g_i\|_{H^{1/2}(\Gamma_i)}\right\}. \quad\square\end{aligned}$$

Theorem S.7 (Full Multigrid Convergence). *If m is chosen large enough so that the kth-level iteration is a contraction with respect to $|\!|\!|\cdot|\!|\!|_{0,k}$ and the parameter r in the full multigrid algorithm is also chosen large enough, then*

$$(S.27)\qquad \begin{aligned} &\|u_k - u_k^*\|_{L^2(\Omega)} + h_k\left(\|u_k - u_k^*\|_k + \|p_k - p_k^*\|_{L^2(\Omega)}\right)\\ &\qquad\le C\,h_k^2\left\{\|f\|_{L^2(\Omega)} + \sum_{i=1}^n \|g_i\|_{H^{1/2}(\Gamma_i)}\right\}.\end{aligned}$$

Here, (u_k^*, p_k^*) is the approximate solution of (1.26) obtained from the full multigrid algorithm.

Proof. By Theorem S.5 we can choose m sufficiently large so that the kth-level iteration ($k = 1, 2, 3, \cdots$) has a contraction number bounded by δ, where $0 < \delta < 1$ is independent of k.

From the full multigrid algorithm, Lemma S.6 and Lemma 3.4 (i), there exists a positive constant C^* such that

$$(S.28)\qquad \begin{aligned} &|\!|\!|(u_k - u_k^*, p_k - p_k^*)|\!|\!|_{0,k}\\ &\le \delta^r\,|\!|\!|(u_k, p_k) - I_{k-1}^k(u_{k-1}^*, p_{k-1}^*)|\!|\!|_{0,k}\\ &\le \delta^r\Big\{|\!|\!|(u_k, p_k) - I_{k-1}^k(u_{k-1}, p_{k-1})|\!|\!|_{0,k}\\ &\qquad + |\!|\!|I_{k-1}^k(u_{k-1} - u_{k-1}^*, p_{k-1} - p_{k-1}^*)|\!|\!|_{0,k}\Big\}\\ &\le C^*\delta^r h_k^2\left\{\|f\|_{L^2(\Omega)} + \sum_{i=1}^n \|g_i\|_{H^{1/2}(\Gamma_i)}\right\}\\ &\qquad + C^*\delta^r|\!|\!|(u_{k-1} - u_{k-1}^*, p_{k-1} - p_{k-1}^*)|\!|\!|_{0,k-1}.\end{aligned}$$

Since the first-level iteration is an exact solver, by iterating (S.28) we obtain

$$
\begin{aligned}
\text{(S.29)}\qquad & \left\| \left(\underset{\sim}{u}_k - \underset{\sim}{u}_k^*, p_k - p_k^*\right) \right\|_{0,k} \\
&\leq \left(C^*\delta^r h_k^2 + (C^*)^2\delta^{2r}h_{k-1}^2 + \cdots + (C^*)^{k-1}\delta^{(k-1)r}h_2^2\right) \\
&\qquad \cdot \left\{ \|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)} + \sum_{i=1}^{n} \|\underset{\sim}{g}_i\|_{\underset{\sim}{H}^{1/2}(\Gamma_i)} \right\} \\
&= \left(C^*\delta^r + 4(C^*)^2\delta^{2r} + \cdots + 4^{k-2}(C^*)^{k-1}\delta^{(k-1)r}\right) h_k^2 \\
&\qquad \cdot \left\{ \|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)} + \sum_{i=1}^{n} \|\underset{\sim}{g}_i\|_{\underset{\sim}{H}^{1/2}(\Gamma_i)} \right\} \\
&\leq \left(\frac{C^*\delta^r}{1 - 4C^*\delta^r}\right) h_k^2 \left\{ \|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)} + \sum_{i=1}^{n} \|\underset{\sim}{g}_i\|_{\underset{\sim}{H}^{1/2}(\Gamma_i)} \right\}
\end{aligned}
$$

if $4C^*\delta^r < 1$, or equivalently, if $r \geq |(\ln 4C^*)/(\ln \delta)|$. Therefore, for such choice of r we have by (3.8) that

$$
\text{(S.30)}\qquad \|\underset{\sim}{u}_k - \underset{\sim}{u}_k^*\|_{\underset{\sim}{L}^2(\Omega)} + h_k \|p_k - p_k^*\|_{L^2(\Omega)} \leq C h_k^2 \left\{ \|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)} + \sum_{i=1}^{n} \|\underset{\sim}{g}_i\|_{\underset{\sim}{H}^{1/2}(\Gamma_i)} \right\}.
$$

The inverse estimate (3.6) then implies

$$
\text{(S.31)}\qquad \|\underset{\sim}{u}_k - \underset{\sim}{u}_k^*\|_k \leq C h_k \left\{ \|\underset{\sim}{f}\|_{\underset{\sim}{L}^2(\Omega)} + \sum_{i=1}^{n} \|\underset{\sim}{g}_i\|_{\underset{\sim}{H}^{1/2}(\Gamma_i)} \right\}. \quad \square
$$